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# THE MATHMATE



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**Mission Statement:** The mission of THE MATHMATE is to feature articles about innovative mathematical classroom practices, important and timely educational issues, pedagogical methods, theoretical findings, significant mathematical ideas, and hands-on classroom activities and disseminate this information to students, educators and administrators.

THE MATHMATE, the official journal of the South Carolina Council of Teachers of Mathematics, is published online two times each year – May and January.

**Submission Requirements:** Submissions for THE MATHMATE should be no more than 15 pages in length not counting cover page, abstract, references, tables, and figures. Submissions of more than 15 pages will be reviewed at the discretion of the editorial board. Submissions should conform to the style specified in the *Publications Manual of the American Psychological Association* (6th ed.). All submissions are to be emailed to [scmathmate@gmail.com](mailto:scmathmate@gmail.com) as attachments with a completed Submission Coversheet as page 1 and the article starting on page 2. The coversheet can be found at <http://scctm.org/The-MathMate>.

Submitted files must be saved as MSWord, RTF, or PDF files. Pictures and diagrams must be saved as separate files and appropriately labeled according to APA style. Copyright information will be sent once an article is reviewed but authors should not submit the same article to another publication while it is in review for THE MATHMATE.

**Submission Deadlines:** Submissions received by November 1 will be considered for the January issue and March 1 for the May issue.

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**Message from the SCCTM President**

Dear SCCTM Members,

The board of SCCTM would like to thank all authors for sharing their knowledge and expertise by writing for THE MATHMATE. We would also like to recognize and thank Chris Duncan in his new role as THE MATHMATE Editor and the reviewers for their careful reviews and consideration of the authors' work. We hope you will consider sharing your expertise with the membership of SCCTM by writing about your successful lessons, activities, and classroom-based strategies. THE MATHMATE serves as a vehicle to connect us and help us learn from one another as we collectively strive to ensure all students have access to high quality mathematics education. Without your submissions, we lose this wonderful opportunity to share and learn from one another.

Sincerely,

Leigh Martin

## **Announcements**

Upcoming Conference Information and Deadlines:

### **SCCTM Fall Conference 2018**

Columbia Metropolitan Convention Center  
November 14 – 16

Speaker Proposal Deadline: June 10

Early Bird Registration Deadline: October 13

[scctm.org/conferences](http://scctm.org/conferences)

Award Nomination Deadlines:

### **Outstanding Contributions to Mathematics Education Award**

Nomination deadline: July 15

[scctm.org/Awards](http://scctm.org/Awards)

### **Richard W. Riley Award**

Nomination deadline: July 15

[scctm.org/Awards](http://scctm.org/Awards)

Scholarship Deadlines:

### **Preservice Scholarship**

Applications deadline: September 15

[scctm.org/scholarships](http://scctm.org/scholarships)

### **Educator's Scholarship**

Application deadline: September 15

[scctm.org/scholarships](http://scctm.org/scholarships)

Membership News:

[Renew your NCTM membership online](#) and designate *South Carolina Council of Teachers of Mathematics* for the affiliate rebate.

If you would like your announcement to appear in the next issue of THE MATHMATE, please email all information to [SCMathMate@gmail.com](mailto:SCMathMate@gmail.com). Announcements will be published at the discretion of THE MATHMATE Editorial Board.

### Message from the Editor

This is my first issue as editor of THE MATHMATE! I've very excited to serve in this role and hope that I can lead the journal as well as the previous editor, Gina Dunn. I need your help, though. Please consider submitting items to the journal. The journal welcomes articles which relate your experience with interesting classroom activities, lessons, and teaching strategies. Articles of a theoretical or research nature are also encouraged. If you are presenting at the upcoming conference in November, please consider creating and submitting a written version of your presentation. While not every SCCTM member will be able to attend the conference, we can all read THE MATHMATE.

In this issue, I've published, with permission, an article from NCTM's journal *Teaching Children Mathematics*. The article, "Inspection Worthy Mistakes: Which? And Why?", helps the reader decide which types of student mistakes make for good class discussions. I chose this article because the ideas it presents are applicable to any math classroom. It should be interesting to any reader. In future issues, I will continue reprinting articles from NCTM journals. The selection process will focus on either general reader interest or on selecting an article which will better round out that particular issue. Many thanks to Dr. Ryan Higgins, our NCTM representative for this idea.

I also would like to thank Jennifer Thorsten, 6 – 12 Mathematics Coordinator for Berkeley County School District, who is serving as Associate Editor for this issue. THE MATHMATE uses a double-blind peer review process in which the editor serves as the intermediary between an author and a reviewer. I submitted an article to this issue and, being the editor, could not maintain that double-blind standard without some help. Jen served as editor for my article by securing a reviewer and helping to usher my article through the revision process.

Finally, for the cover of this issue, I chose a photograph of the Columbia skyline to highlight that our fall conference is returning to Columbia this year. It will be held at the Columbia Metropolitan Convention Center on November 14, 15, and 16. This is the first time the fall conference has been held in Columbia since 2009!

I hope you enjoy this issue. Please send comments, questions, or concerns to the journal's email address: [scmathmate@gmail.com](mailto:scmathmate@gmail.com)

Enjoy!

Chris

## Discovery of Power and Exponential Relationships: An Application of Logarithms

Chris Duncan  
Lander University

### Abstract

This article discusses log-log and semi-log graphing techniques which are useful in determining power or exponential relationships between two variables. These graphs are a common application of logarithms in various scientific and engineering disciplines and their use in a math classroom reinforces the properties of logarithms and can be used in various active-learning, in-class lab type experiences. Example applications related to Cantilever Beams and the temperature of a liquid are discussed.

### Introduction

Hands-on modeling exercises are good for students of mathematics. Specific applications from science or other stem fields are particularly valuable and meet NCTM process standards by allowing students to make connections between mathematics and other subjects. Such activities align with the South Carolina College- and Career-Ready Standards for Mathematics process standards by giving students an opportunity to reason contextually as well as to connect mathematics and the real-world through modeling. In this paper, a few such activities are described, and the mathematics involved would be appropriate for a Pre-Calculus class.

The key objective of these activities is discovering the nature of a relationship between two quantities. The method employed here has the students take the logarithm of one or both quantities, graph the resulting data, and, assuming the result is approximately linear, fit a line to the transformed data. The coefficients of the line provide information about the relationship between the two quantities. Graphing calculators could be used to skip these steps altogether, and this may be appropriate for students who have not studied logarithms. For the Pre-Calculus student, the method described here reinforces the properties of logarithms and demonstrates a practical use of those properties. The activities could be used to introduce nonlinear regression models by having the student compare his or her answer using the method described here with the calculator's answer.

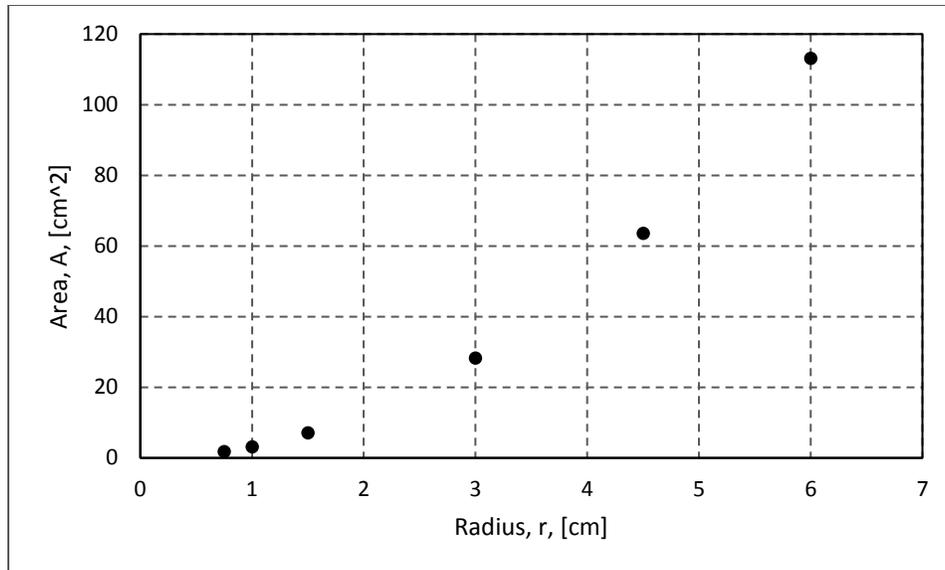
### Mathematical Background

#### Power Relationships

We say that two variables,  $x$  and  $y$ , have a power relationship, if  $y = b x^m$ , for some constants  $b$  and  $m$ . For example,  $A = \pi r^2$  says that the area of a circle is a power function of the radius. Here,  $b = \pi$  and  $m = 2$ . Similarly, the volume of a cube is a power function of the cube's side length with  $b = 1$  and  $m = 3$ .

In many real-world problems, we may have knowledge or a suspicion that a relationship between two quantities exhibits a power relationship, but we may not know  $b$  and  $m$ . We can investigate the relationship by first collecting data, then applying a logarithm to one or both variables, and, finally, checking to see if the new relationship is linear.

For example, suppose the equation which relates the area of a circle to its radius were unknown to us. We could start to explore the relationship by collecting data on circles. Suppose the following graph represents data collected on six circles.



We now ask the following two questions.

- Do the data illustrate a power relationship between  $A$  and  $r$ ? (1)  
 If so, what are the values of  $m$  and  $b$ ? (2)

To answer these questions, note that *if* there is a power function which fits this data, then there are constants  $m$  and  $b$  such that  $A = br^m$ . Taking the logarithm of both sides and simplifying yields the following.

$$\begin{aligned} A &= br^m \\ \log A &= \log(br^m) \\ \log A &= \log b + \log r^m \\ \log A &= \log b + m \log r \end{aligned}$$

This illustrates that  $\log A$  is a linear function of  $\log r$ . This situation may be more clear if we let  $Y = \log A$ ,  $X = \log r$ , and  $B = \log b$ , then the last equation is

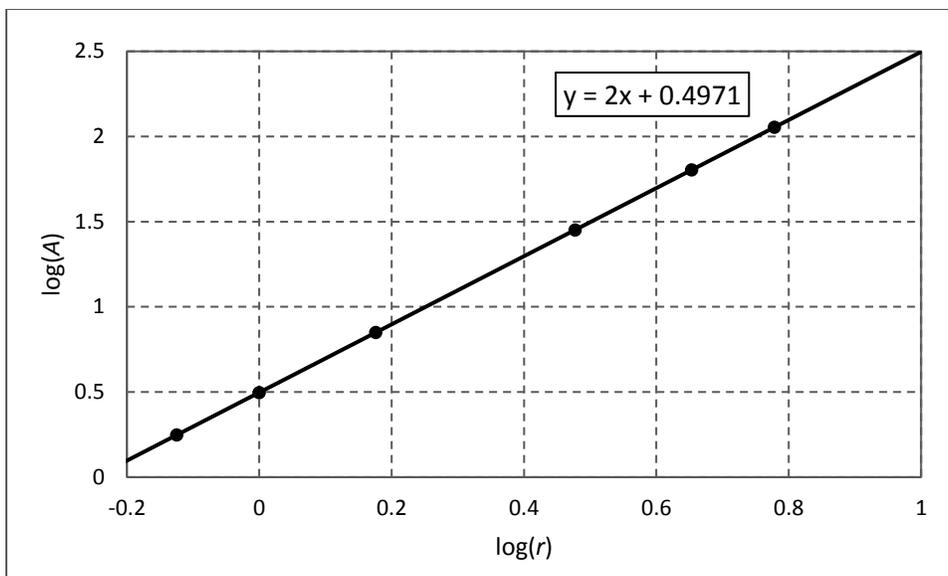
$$Y = B + mX.$$

So, clearly,  $Y$  is a linear function of  $X$ . Of course, using the identities for logarithms in the reverse order as above, and the fact that the logarithm is one-to-one, shows that if  $\log A$  is a linear function of  $\log r$ , then  $A$  is a power function of  $r$ .

This calculation converts questions (1) and (2) to the following

- Is  $\log A$  a linear function of  $\log r$ ? (3)  
 If so, what are the slope and y-intercept of the line? (4)

The answers can be determined by creating a log-log plot of  $\log A$  against  $\log r$ , and then, if the association is linear, fitting a line to the data. The following plot illustrates this so called log-log plot.



The data clearly suggest a linear trend is appropriate. Further,  $m = 2$ , and  $B = 0.4971$ , approximately. Since  $B = \log b$ , we have  $b = 10^B = 10^{0.4971} = 3.1412$  which is pretty close to the actual value,  $\pi$ .

It should be noted that the exact value for the exponent was obtained in this example because the data was not actually collected. Instead the author, for illustration, used the formula for the area of a circle in order to generate the data. If one were to actually attempt to approximate the area and radius of several circles, there would be errors in the measurements which would result in only approximate values for the two constants. This can be seen in the activities describe later.

Note that if students have not been exposed to the line of best fit, they could instead draw a reasonable line that fits the data and then determine the equation of the line that was drawn. Alternatively, graphing calculators provide the line of best fit.

### Exponential Relationships

An exponential relationship can be expressed in a variety of bases, but in this discussion we will use the natural base,  $e$ . We will consider exponential functions of the form  $y = b e^{mx}$ , for constants  $b$  and  $m$ . The goal is to determine if some collected data represent such a relationship, and, if so, the values of  $b$  and  $m$ .

Following the same procedure as in the previous section, we take a logarithm of both sides and use the properties of logarithms to simplify. It would seem prudent to use the natural logarithm since that is the base we have chosen. However, it is common to use base 10 because that is the base typically used on logarithmically scaled axes. We will use base 10 here, but, again, base  $e$  or any other base is just as appropriate.

$$\begin{aligned}
 y &= b e^{mx} \\
 \log y &= \log(b e^{mx}) \\
 \log y &= \log b + \log e^{mx} \\
 \log y &= \log b + m \log e x
 \end{aligned}$$

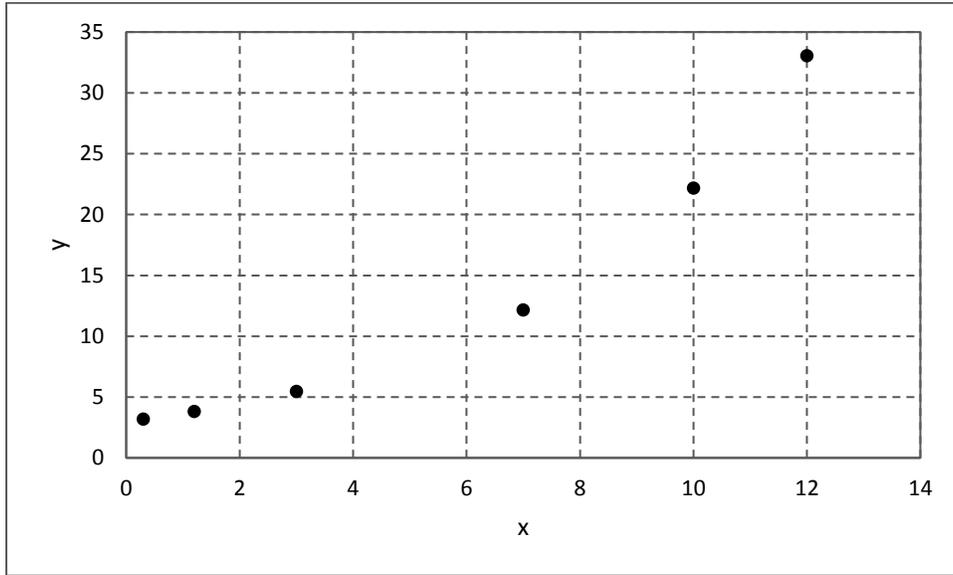
This shows that *if*  $y$  is exponential then  $\log y$  is a linear function of  $x$ . In fact, if  $Y = \log y$ ,  $B = \log b$  and  $M = m \log e$ , then

$$Y = B + Mx.$$

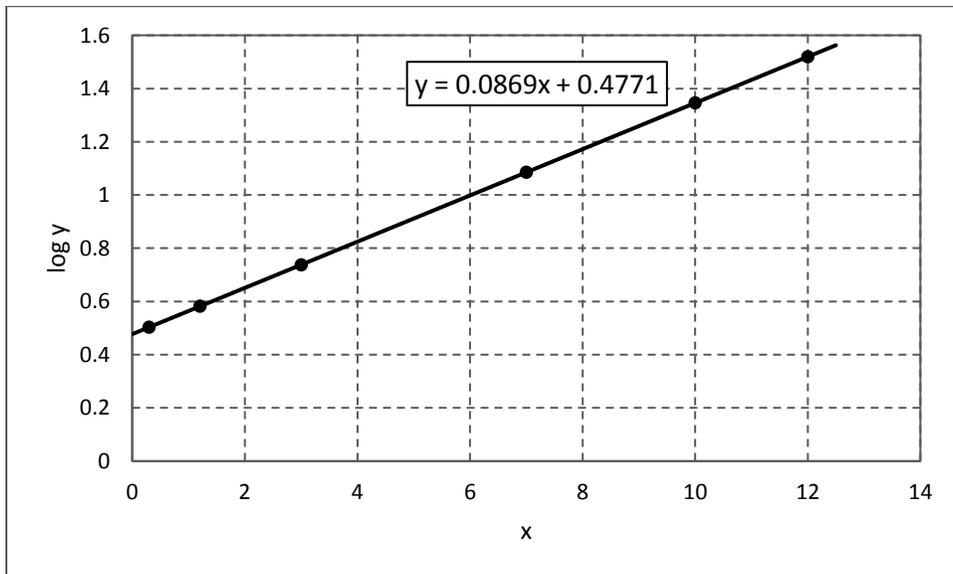
The converse is true as well. If  $Y$  and  $x$  have a linear relationship, then  $Y$  is an exponential function of  $x$ .

If we create a semi-log plot of  $\log y$  against  $x$ , and see a linear relationship, then there are constants  $b$  and  $m$  such that  $y = b e^{mx}$ . Further, we calculate  $b = 10^B$  and  $m = M/\log e$ .

For example, consider the plot below of the following fictitious data: (0.3, 3.18551), (1.2, 3.813747), (3, 5.466356), (7, 12.1656), (10, 22.16717), (12, 33.06953).



We ask questions similar to (1) and (2), above. Does this represent a relationship of the form  $y = b e^{mx}$ , and if so what are the values of  $b$  and  $m$ ? A plot of  $\log y$  against  $x$  answers the first question, and a couple of calculations determine  $b$  and  $m$ .



We have  $b = 10^B = 10^{0.4771} = 3$  and  $m = 0.0869/\log e = 0.2$ . Thus,  $y = 3 e^{0.2x}$ . A check of the original data shows that this is in fact the relationship present in that data.

### Student Activities

The remainder of this article is devoted to two lab activities that can be discussed and implemented in the classroom

#### Cantilevered Beams

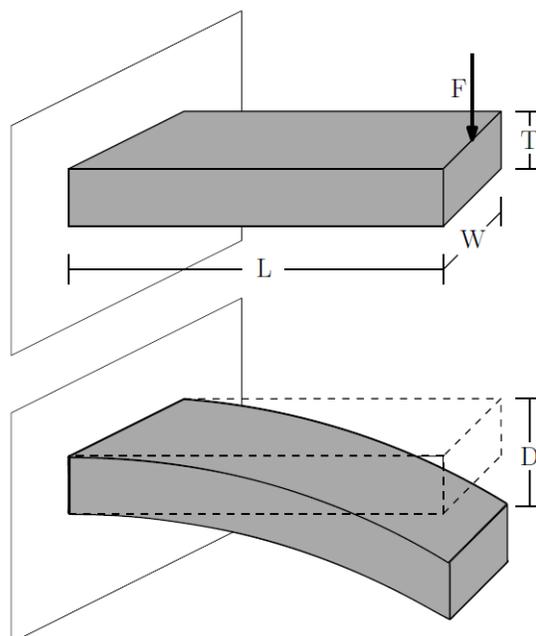
The discussion is divided into 4 sections. First, some background information about cantilevered beams is described, then the necessary materials and set-up is given. Then data collection and analysis is discussed. Finally, an example problem concerning diving boards that can be posed to students is given.

#### *Background Information:*

A cantilevered beam is a beam that is fixed on one end and free on the other. These beams occur often in decks, porches, and other engineering applications. A central concern is the amount of deflection that the beam will experience when a load (i.e., force) is applied to the beam. If a load,  $F$ , is applied to the free end of the beam, the deflection,  $D$ , at that end is given by the following.

$$D = \frac{4 F L^3}{E W T^3}$$

Where,  $L$  is the length of the beam,  $W$  is its width,  $T$  is its thickness, and  $E$  is the Young's Modulus of the beam which is a measure of the elasticity or stiffness of the material from which the beam is constructed. The units of measurement for these variables should be consistent. All lengths should be measured in the same units, for example. In this paper we will measure lengths in centimeters and the force in Newtons.



In the activity described below, the student will focus on discovering the exponents of  $F$  and  $L$ . The other variables will not be changing and can be considered constants. Thus, the goal for the students is to discover that the deflection satisfies

$$D = k F^{b_1} L^{b_2} \tag{5}$$

with  $b_1 = 1$  and  $b_2 = 3$  when all other aspects of the beam are fixed.

By varying only the load,  $F$ , and measuring the deflection, the student can explore the relationship between  $D$  and  $F$ . The goal is to determine if that relationship is a power relationship, i.e.  $D = m F^b$ , and if so the value of the exponent. Note that the value of  $m$  represents the combined effects of all of the other physical aspects of the beam.



Figure 1: The beam is attached to a table or desk with the C-clamp. The bag holds weights or marbles, and the meter stick is used to measure the amount of the deflection. In this example the length of the beam was varied but force (number of marbles in the bag) was held fixed.

In a similar manner, if the deflection is measured while only the length of the beam is changed, then the student can consider the relationship between  $D$  and  $L$ . Is that relationship of the form  $D = m L^b$ , and, if so, what is the value of  $b$ ?

*Set up and materials:*

For the activity, each student of group of students will need a meter stick, a plastic bag, a C-clamp, an S-shaped hook, an aluminum strip, and either weights or numerous items that are approximately uniform in weight. The clamp, hook, and aluminum strip can be purchased at a big-box home improvement store for about \$4. See Figure 1 for an illustration of how to set up these materials.

*Data Collection and Analysis:*

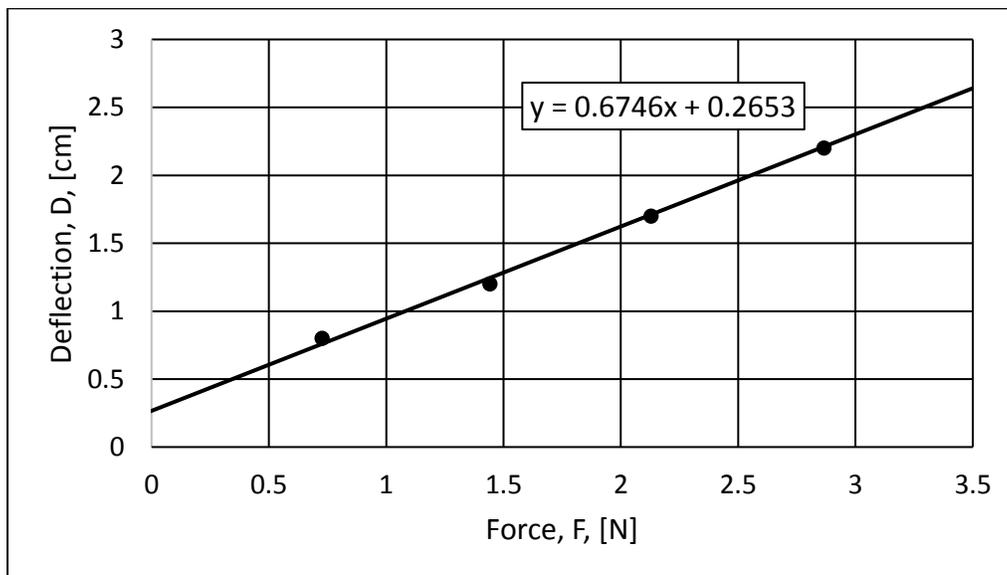
As stated above, the data collection is divided into two parts. First, the student will determine the relationship between the deflection and the applied force. Then, separately, the relationship between the deflection and the length will be explored.

First we consider the relationship between deflection and the force applied. The length is not changed.

Mass (g)	Force (N)	Deflection (cm)
74	0.73	0.7
147	1.44	1.1
217	2.13	1.6
292	2.86	2.1
362	3.55	2.6

The author used a kitchen scale to weigh the marbles. Like most modern scales this one gave readings in units of mass. The first column of the table gives the readings from the scale. Technically, we are modeling the effect of force (weight in this case) not mass. Force is measured in Newtons, and near the surface of the Earth, one may calculate the weight corresponding to a mass in grams by dividing the mass by 1000 (convert to kg) and then multiplying by 9.8 meters per second squared.

The graph below shows the Deflection plotted against the weight of the marbles. The relationship appears to be linear on unscaled axes. So the deflection is a linear function of weight. Since the relationship is linear, we conclude that  $b_1$  in equation (5) has value 1.



Next, we illustrate the relationship between the deflection and length of the beam. In order to estimate the exponent of  $L$  in equation (5), the student should hold  $W$ ,  $F$ , and  $T$  fixed so that equation (5) becomes  $D = KL^{b_2}$ . The student should pick a fixed load that will be applied to various lengths of beams. The length can be varied by simply loosening the c-clamp and sliding the beam in or out.

Example data are below.

L [cm]	D [cm]	log(L)	log(D)
14	0.8	1.15	-0.10
23.5	2.9	1.37	0.46
27.2	4.7	1.43	0.67
34.4	8.6	1.54	0.93
41	13.6	1.61	1.13

The relationship between  $L$  and  $D$  is clearly not linear. However, a plot of  $\log(D)$  against  $\log(L)$  does appear linear. The line of best fit has slope 2.65. Thus, we estimate that  $c$  in equation (5) has value 2.65. This is somewhat less than the theoretical value of 3.

If one wants to attempt to have the students estimate the exponent associated with the thickness, a possible approach is to fix the length and load, and then measure the deflection for a variety of thickness. Beams of varying thicknesses could be constructed by stacking multiple beams on top of each other to form a thicker beam. Also, if several widths of aluminum strips are available, then the exponent on  $W$  can be explored. The author has not tried either of these with a class.

*Example Problem:*

The section concludes with a possible application problem involving diving board design. Diving boards are not exactly like the cantilevered beams presented here, but can be modeled as such and do provide a context which is familiar to most students. This problem should be posed prior to the activity. Then using the results of the activity the students can answer the question.

Suppose a 575 N woman (about 130 pounds) is constructing a diving board for her pool. She would like for the beam to deflect 4 cm when she stands on the end. She constructs the board so that it is length is 185 cm. However, when she stands on it, the board only deflects 2.5 cm. She decides that the board should be longer so that it will deflect further. What length will give the desired deflection of 4 cm?

The solution involves two steps. We are asked to find a value of  $L$  given values for  $F$  and  $D$ . However, using the results from the example data above, we have

$$D = k F L^{2.65}$$

The value of  $k$ , which encodes information about the elasticity, thickness, and width of the board, needs to be determined. The initial attempt of a 185 cm long board giving a deflection of 2.5 cm allows us to determine  $k$  using our calculated formula above. Substituting  $D = 2.5$ ,  $F = 575$ ,  $L = 185$ , and solving for  $k$  gives an approximate value of  $k = 4.3 \times 10^{-9}$ . Thus,

$$D = (4.3 \times 10^{-9}) F L^{2.65}$$

Next, we can determine the correct length by substituting the desired deflection of 4 cm, and the force of 575 Newtons and solving for  $L$ . This results in a board length of about 220 cm.

Another question using a diving board could involve a maximum allowable force given a fixed length and the maximum possible deflection before the board breaks. Alternatively, one can ask for a given force, what is the maximum possible length before the board fails.

Cooling Liquid

Next an example of an exponential relationship is presented. As students know from everyday experience the temperature of an object approaches the temperature of its surroundings. That approach is exponential in nature as can be seen in the following activity.

*Background Information:*

Newton's law of cooling states that the rate of change of the temperature of a body is proportional to difference between the ambient temperature and the body's temperature. This implies that temperature will approach the ambient temperature exponentially fast. In other words, the difference between the temperature and the ambient temperature is exponential. So,

$$T - A = be^{mt} \tag{6}$$

with  $T$  representing the temperature of the body,  $A$  standing for the ambient temperature,  $t$  representing the time, and  $b$  and  $m$  some constants.

An activity related to this topic would have the students measure the temperature of a hot (or cold) cup of water at various times. Then following the semi-log procedure discussed above, estimating the value

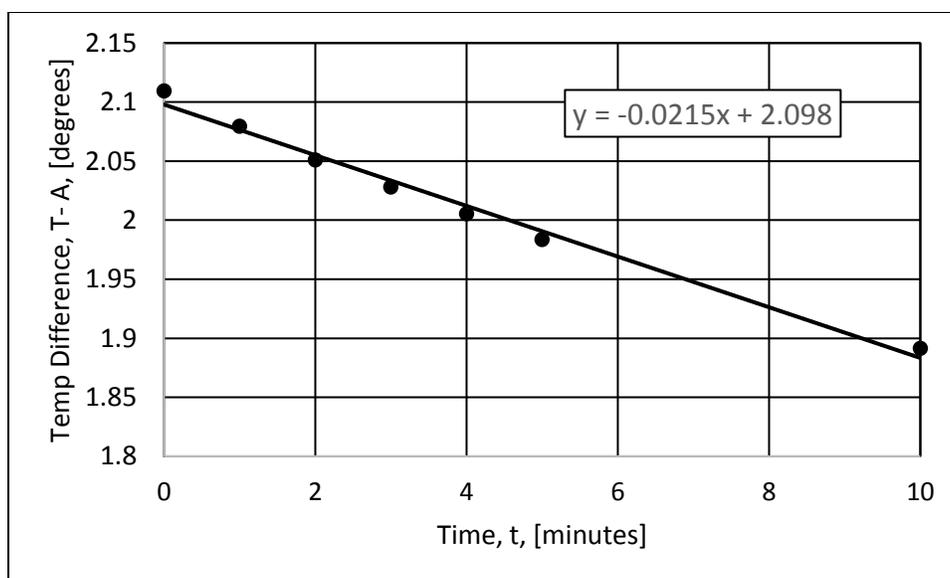
of the constants using linear regression. As suggested by equation (6), the difference between the water and the ambient temperature is the appropriate dependent variable rather than the temperature itself. This is important since there is no identity for the logarithm of a sum. So, if the student sets the problem up as  $T = A + be^{mt}$ , and then takes the logarithm of both sides, he or she will arrive at a roadblock since  $\log T = \log(A + be^{mt})$  does not express  $\log T$  as a linear function of  $t$ .

*Data Collection and Analysis:*

As an example, a cup of water initially at 198.2 degrees Fahrenheit sat in a room in which the ambient temperature was recorded to be 69 degrees. The temperature of the cup of hot water was measured at various times as in the following table.

Time, t	0	1	2	3	4	5	10
Temp, T	198.2	189.6	182	176.2	170.8	165.8	147.4
T-A	128.7	120.1	112.5	106.7	101.3	96.3	77.9

In the following graph, we can see that on the semi-log plot, the relationship between  $T - A$  and  $t$  appears linear with  $B = 2.098$  and  $M = -0.0215$ .



So,  $b = 10^B = 125.3$  and  $m = M/\log e = -0.04951$ . Substituting these values in equation (6) yields the following.

$$T = 69 + 125.3e^{-0.04951 t}$$

This type of activity could be extended to compare the results using different liquids (water, oil, etc.) and or different types (insulated or not) or shapes (narrow or wide opening) of cups. A potential question then would be how the values of  $m$  or  $b$  depend on the different scenarios, and how to interpret the meaning or size of the quantities in the physical, real-world context.

## Conclusion

We have given two examples of in-class activities that illustrate how logarithms are useful for discovering power and exponential relationships in STEM settings. While the lab portion promotes student engagement and understanding of the application being modeled, it is not strictly necessary. If time or materials are in short supply, a teacher could supply the data and have the student discover the relationships from that starting point.

Other examples of power relationships that can be explored include

1. If an object is dropped from rest, the relationship between the distance it has fallen and the time since it was dropped is a power function with exponent approximately 2.
2. Among the planets of the solar system, the orbital period (year length) is a power function of the average distance from the sun. The exponent is  $3/2$ .
3. The Square-Cube Law is concerned with the relationship between an object's surface area and its volume as the object grows. A simple example is a cube. The volume is a power function of the area with an exponent of  $3/2$ . The same is true of a sphere. These examples have the added benefit of being able to verify the results using the two formulas and some algebra.

Other examples of exponential relationships that can be explored include

1. The balance in an account that earns interest is an exponential function of the amount of time since the money was deposited.
2. The number of bacteria depends exponentially on how long they have been dividing.
3. Radioactive decay is exponential in time. Carbon dating uses the decay of radiocarbon to determine the age of a bone.
4. The cooling liquid example above could easily be recast as a time of death investigation in a whodunit style exploration.

## References

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*, NCTM: Reston, VA

South Carolina Department of Education (2015). *South Carolina College- and Career-Ready Standards for Mathematics*. SCDE: Columbia, SC.

Gere, James and Goodno, Barry. (2012). *Mechanics of Materials*, 8<sup>th</sup> ed. Cengage: Stamford, CT

Edwards, C. Henry and Penney, David. (2008). *Elementary Differential Equations*, 6<sup>th</sup> ed. Pearson: Upper Saddle River, NJ

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## An Examination of Pre-service Teachers' Efficacy

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### Abstract

A sequential explanatory design was utilized to determine the general and personal teaching efficacy of K-8 pre-service teachers developing through their teacher-preparation programs. Results suggested participants leaned toward innovative beliefs related to teaching mathematics and the relevance of mathematics and claimed to be innovative in their practices and have high personal teaching efficacy; however, participants' general beliefs in the efficacy of mathematics teaching were consistently the lowest. Overall, findings convey a narrative of consequential overconfidence.

### Introduction

In the Lowcountry in 2020, a predicted 26,000 jobs will be available; many of the highest paid, STEM-related positions likely will be filled by non-SC residents based on inadequate K-16 mathematics preparation (Avalanche, 2016; Pan, 2017). The National Council of Teachers of Mathematics (NCTM) has been working for decades to change traditional instruction, encouraging the implementation of innovative pedagogical practices – emphasizing communication, connections, and problem solving – that result in deep mathematical understanding for all children so they are, in fact, prepared for such STEM positions upon entry into the workforce (NCTM, 2000).

Necessary for such a shift – and challenging for the Lowcountry and nation to date – is the recruitment and retention of teachers who implement such excellent, equitable practices (Avalanche, 2016; Will, 2016). To consider is teachers' confidence, or efficacy, in the profession in general and in their own personal teaching: “efficacy beliefs can influence career aspiration and longevity” (Siwatu & Chesnut, 2015, p. 214). Bandura (1997) describes simply, “People avoid activities ... they believe exceed their capabilities” (p. 160).

Efficacy beliefs influence and predict instructional decisions and actions (Buehl & Beck, 2015) and, subsequently, student learning (Bikkar, Beamer, & Lundberg, 1993; Richard & Liang, 2008). High efficacy has been linked to greater willingness to adopt innovative pedagogical beliefs and implement innovative practices (Jerald, 2007, in Protheroe, 2008; Cross Francis et al., 2015) and, in turn, to “students who learn” (Shaughnessy, 2004, in Protheroe, 2008, p. 43). To increase efficacy requires successful instructional experiences that yield gains in student achievement (Buehl & Beck, 2015); therefore, increased efficacy and increased student learning feed one another.

While most research suggests that high efficacy is beneficial, Wheatley (2002) reported that individuals with efficacy doubts may be more motivated to “learn and improve,” increasingly implementing innovative practices (in Siwatu & Chesnut, 2015, p. 217). Too, those with high efficacy may have false confidence: “...a false sense ... sets [teachers] up for disillusionment and burnout. ... [Or] blam[ing] students or whomever for their struggles, when the real problem is overconfidence” (p.217).

Certainly pre-service teachers (hereafter, “PTs”) are important to consider related to recruitment and retention, as “some of the most powerful influences on the development of teacher efficacy are

...experiences during student teaching” (Hoy, 2000, in Protheroe, 2008, p. 43). Related specifically to their personal efficacy in teaching and doing mathematics, PTs reportedly lack confidence and are highly anxious (Brand & Wilkins, 2007; Liang & Richardson, 2009; Bursal & Paznokas, 2006, all in Rethlefsen & Park, 2011).

While mathematics teacher educators are working to improve K-12 PTs’ efficacy beliefs necessary for the cultural transformation from traditional to innovative pedagogical practices, contributions by educators at all levels are necessary for this cultural shift to occur. In particular, K-12 mathematics educators and curriculum specialists can work toward helping K-12 students – a subset of whom will feed into teacher-preparation programs – develop confidence in their mathematical abilities, beliefs that mathematics is relevant and meaningful, and beliefs that all people, themselves included, can do math. This confidence and these beliefs serve as the foundation to both personal and general teaching efficacy beliefs.

Given the implications of efficacy beliefs, particularly related to PTs, this study is guided by two sets of questions. Set #1: What are PTs’ entering, developing, and exiting beliefs regarding the general power of mathematics teaching [MTOE]? How do these beliefs change throughout their teacher-preparation programs? How do these beliefs relate to/vary from beliefs about what and how mathematics should be taught [BT] and beliefs about mathematical relevance [BRW]<sup>1</sup>? Set #2: What are PTs’ claims about the power of their own mathematics teaching ability [PMTE] upon completion of their preparation programs? How do these claims relate to/vary from their claims about their own instructional practices [P] and to their beliefs about the general power of teaching [MTOE]? Understanding the entering efficacy beliefs of pre-service teachers, how these beliefs develop over time, and how they relate to other mathematical beliefs can highlight for K-16 mathematics educators and teacher educators areas to target toward positively influencing efficacy beliefs among educators and, subsequently, confidence and achievement in mathematical ability among students.

### **Methodology**

Eighty-five K-8 PTs<sup>2</sup> entering their four-semester, undergraduate teacher-preparation programs within a public, liberal arts and science college were asked to complete three survey iterations: A1, A2, and A3. Eighty-three PTs consented and completed a hard copy of A1 administered upon entry into their programs in August 2011. Sixty-four PTs completed a hard copy of A2 in May 2012 at the completion of their second mathematics-education (hereafter, “math ed”) course. The first course developed mathematical content knowledge and the second pedagogical practices. Mathematics educators provided explicit experiences to promote increased confidence and buy-in regarding innovative practices aligned with those espoused by NCTM. Attrition between A1 and A2 was likely due to the fact that the second math ed course in which A2 was administered was not mandatory for non-mathematics middle-level education majors. Forty-four PTs completed an electronic version of A3 at the completion of their programs, following a one-semester clinical internship. Attrition in A3 was due to the fact that eight PTs switched majors or left the college and likely to the loss of face-to-face interaction given A3’s electronic administration.

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<sup>1</sup> For the scope of this paper, general comparisons were made. Further exploration into the direct relationships between specific BT, BRW, and MTOE beliefs and between specific P and PMTE claims would be extremely interesting

<sup>2</sup> Given no statistical difference in responses between early-childhood, elementary, and middle-level education majors, the sample of K-8 PTs was examined in aggregate, allowing for a larger sample size.

The survey<sup>3</sup> consisted of open-ended and Likert items scaled from one to five with one being strongly disagree and five strongly agree (see Appendix A). Likert items written in third person were used to gain information about general normative beliefs. On A3 only, items written in first person were included to gain information on what PTs claimed about their personal pedagogical practices following sustained teaching during their internships. For most items, a value of one was equivalent to highly traditional and a value of five was equivalent to highly innovative. Items which asked about innovative beliefs in a negative manner or traditional beliefs in a positive manner were reverse coded.

Items were combined to form subscales based on face validity and verified through reliability factor analyses. The general-normative-beliefs subscales are What and How Mathematics Should be Taught [BT] (#1-5,7-10,12-15); Mathematics in the Real World [BRW] (#1-4,6); and Efficacy in Mathematics Teaching [MTOE] (#1,4,5) (Enochs et al., 2000, p. 195). The personal-claims subscales are My Personal Efficacy in Mathematics Teaching [PMTE] (#2,3,6,7) (p. 195) and My Teaching Practices [P] (#1-4,6-17). This paper focuses largely on MTOE items and subscale, 60.5% reliable, and PMTE items and subscale (73.1% reliable), with comparisons made between these and other subscales<sup>4</sup>.

Descriptive statistics on Likert items were calculated and paired t-tests were conducted to describe MTOE beliefs and PMTE claims and to highlight differences between other types of beliefs (BT, BRW, and/or P) at a given time: A1, A2, and A3. As stated, attrition was experienced between iterations<sup>5</sup>, causing pause when making comparisons between iterations. Dismissing responses except those that came from PTs who responded to all iterations would diminish the study's power and have implications related to the representation of the cohort; however, the use of all data might skew results. Therefore, an analysis of patterns of non-response was conducted to determine the appropriate treatment for missing data (Pigott, 2001; Peugh & Enders, 2004). Paired-t-tests were conducted to identify differences between iteration responses.

Open-ended responses were analyzed. Initially, responses within an iteration were analyzed in aggregate to identify categories (Green et al., 2007). Frequency percentages were obtained through counting coding

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<sup>3</sup> Adapted from Enoch, Smith, & Huinker (2000), "Establishing factorial validity of the Mathematics Teaching Efficacy Beliefs Instrument" in *School Science and Mathematics*. Reprinted by permission of Larry Enoch. The acronym "MTOE" stands for the "mathematics teaching outcome expectancy" and "PMTE" for "personal mathematics teaching efficacy" (2000, p. 195).

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Adapted from Showalter (2005), *Teacher interview protocol in the doctoral dissertation: The effect of middle school teachers' mathematics teaching self-efficacy beliefs on their students' attitudes toward mathematics*. Reprinted by permission of Betsy Showalter.

<sup>4</sup> Details related to the other subscales can be found in Lloyd and Howell (accepted for publication in *The Mathematics Enthusiast* vol. 15, no.3 [July 2018]).

<sup>5</sup> Thirty-six (42%) PTs completed all three iterations (A1, A2, and A3), 26 (31%) completed A1 and A2, six (7%) completed A1 and A3, 15 (18%) completed A1 only, and two (2%) completed A2 only.

(Miles & Huberman, 1994). Then, individual PT's responses were examined over time (looking at responses to a given question on A1, A2, and/or A3), individual's belief changes were categorized, and themes among categorizations of change were identified and reported in aggregate once again. Findings were compared to the Likert data analysis to determine if they were supportive or contradictory. Open-ended response and Likert-item analyses were used together to address how PTs' efficacy beliefs transformed.

## Results

### Set #1: Efficacy in Teaching Mathematics [MTOE]

At the beginning of their preparation programs, PTs' MTOE beliefs were significantly less innovative than their corresponding beliefs about What and How Mathematics Should be Taught [BT] and Mathematics in the Real World [BRW] (see Tables 1 & 4). They remained significantly less innovative following math-ed coursework (see Tables 2 & 4) and following internships (see Tables 3 & 4). On every iteration, the most traditional MTOE belief was that if students are underachieving in mathematics, it is most likely due to ineffective teaching (V.4meanA1=2.87, %traditionalA1=41, see Table 1; meanA2=3.11, %traditionalA2=36, see Table 2; meanA3=3.14, %traditionalA1=28, see Table 3). It consistently had the third lowest mean among all BT, BRW, and MTOE items. The most innovative MTOE belief was that the inadequacy of a student's mathematics background can be overcome by good teaching (V.5meanA1=3.90, %innovativeA1=80, see Table 1; meanA2=4.05, %innovativeA2=84, see Table 2; meanA3=4.05, %traditionalA1=84, see Table 3).

Such results imply that PTs credit teaching for "overcoming" but do not discredit teaching for "inadequacy." The open-ended responses supported these findings (see Appendix B). On A2, only two of the 64 PTs indicated that how the teacher teaches "is the most important part of instruction" and that "instruction is what you as teachers make it." They saw the teacher as the most important factor to student achievement through innovative instruction. The majority of the remaining responses, however, were consistent with the Likert data, revealing persistent views related to the locus of control for student achievement or failure. That is, while they acknowledged that teachers may be able to assist in student learning, issues beyond teachers' control negatively influence learning outcomes. One PT writes, "I feel that, while the teacher plays a huge role, it's ultimately the student who succeeds or fails." Others highlighted the influences of parents and the content in negatively impacting student achievement: "...having parents who say 'We're not great at math. We don't expect you to be either.'"

Similarly, when asked if students were excited about mathematics, most PTs who responded that students were excited often credited the teacher: "... I try to keep the classroom fun [emphasis added]." Those that claimed that students were not excited attributed this to influences outside of their influence such as to parental and cultural beliefs: "Kids learn to hate math by seeing it through media, family, friends, etc." T-tests and frequencies conducted post-imputation (n=68) revealed the following on how MTOE beliefs changed throughout preparation programs (see Tables 5 & 6; Figures 1 & 2): (1) Consistently high means and high percentages of innovative responses throughout preparation regarding the ability of good teaching to overcome a student's inadequate mathematics background (MTOE V.5); (2) significant increases in innovative responses at the completion of programs regarding the positive impact of teaching

effort on improved student achievement (MTOE V.1); and (3) consistent means on MTOE V.4, with drops in both high and low responses and an increase in neutral responses on A3, implying that by the end of their programs, a majority of PTs had become more neutral in their stances on teacher impact on student achievement, with a quarter still “blaming” forces beyond their control for underachievement (as corroborated in the open-ended responses on A3). Too, whether using post-imputation data (n=68), all of the data with varying n values (nA1=83; nA2=64; nA3=44), or only the data from those PTs that took all three surveys (n=36; see Figure 3), though MTOE beliefs in aggregate developed toward being more innovative (both shown in increased means and frequencies), they consistently remained the least innovative subscale, particularly because of the belief that underachievement is not due to ineffective teaching (MTOE V.4).

**Table 1: Entering Survey [A1] Subscale, Total, and Individual *Efficacy in Mathematics Teaching* [MTOE] Item Means, Standard Deviations, and Frequency Percentages (n~83)**

	Mean	SD	% Leaning Traditional (Response of 1 or 2)	% Neutral (Response of 3)	% Leaning Innovative (Response of 4 or 5)
<b>A1 Subscales and Total:</b>					
<b>BT_A1_Subscale</b>	3.48	.27	20	20	60
<b>BRW_A1_Subscale</b>	3.81	.55	8	19	72
<b>MTOE_A1_Subscale</b>	<b>3.30</b>	.70	<b>24</b>	28	48
<b>A1_Total</b>	3.54	.30	18	21	61
<b>A1 Individual Survey Items:</b>					
<b>MTOE V.1. “When a student does better than usual in math, it is often because the teacher exerted a little extra effort”</b>	3.13	.95	25	<b>41</b>	34
<b>MTOE V.4. “If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching.”</b>	2.87	.96	<b>41</b>	29	30
<b>MTOE V.5. “The inadequacy of a student’s mathematics background can be overcome by good teaching”</b>	3.90	.76	7	12	<b>80</b>

Note: Percentages reported are the percentage related to the given item, subscale, or total.

**Table 2: Following Math-Ed Coursework Survey [A2] Subscale, Total, and Individual *Efficacy in Mathematics Teaching* [MTOE] Item Means, Standard Deviations, and Frequency Percentages (n~64)**

	Mean	SD	% Leaning Traditional (Response of 1 or 2)	% Neutral (Response of 3)	% Leaning Innovative (Response of 4 or 5)
<b>A2 Subscales and Total:</b>					
<b>BT_A2_Subscale</b>	3.89	.29	14	14	72
<b>BRW_A2_Subscale</b>	4.17	.52	3	12	85
<b>MTOE_A2_Subscale</b>	<b>3.44</b>	.70	23	19	57
<b>A2_Total</b>	3.89	.31	12	14	73
<b>A2 Individual Survey Items:</b>					
<b>MTOE V.1. “When a student does better than usual in math, it is often because the teacher exerted a little extra effort”</b>	3.20	.96	26	28	46
<b>MTOE V.4. “If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching.”</b>	3.11	1.06	36	22	42
<b>MTOE V.5. “The inadequacy of a student’s mathematics background can be overcome by good teaching”</b>	4.05	.83	8	8	84

Note: Percentages reported are the percentage related to the given item, subscale, or total.

**Table 3: Completion of Program Survey [A3] Subscale, Total, and Individual *Efficacy in Mathematics Teaching* [MTOE] Item Means, Standard Deviations, and Frequency Percentages (n~44)**

	Mean	SD	% Leaning Traditional (Response of 1 or 2)	% Neutral (Response of 3)	% Leaning Innovative (Response of 4 or 5)
<b>A3 Subscales and Total:</b>					
BT_A3_Subscale	3.79	.36	15	17	68
BRW_A3_Subscale	4.28	.49	3	7	90
MTOE_A3_Subscale	<b>3.58</b>	.60	14	29	<b>57</b>
P_A3_Subscale	3.83	.45	14	16	70
PMTE_A3_Subscale	3.90	.58	5	20	<b>75</b>
A3_Total	3.84	.32	12	16	71
<b>A3 Individual Survey Items:</b>					
MTOE V.1. “When a student does better than usual in math, it is often because the teacher exerted a little extra effort”	3.56	.88	14	28	58
MTOE V.4. “If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching.”	3.14	.97	28	42	<b>30</b>
MTOE V.5. “The inadequacy of a student’s mathematics background can be overcome by good teaching”	4.05	.62	0	16	<b>84</b>
PMTE V.2. “I am continuously finding better ways to teach mathematics.”	<b>4.19</b>	<b>.66</b>	0	14	<b>86</b>
PMTE V.3. “I know the steps to teach mathematics concepts effectively.”	3.70	.74	5	33	63
PMTE V.6. “When a student has difficulty understanding a math concept, I am usually at a loss as to how to help the students understand it better.”*	3.95	.82	9	7	<b>83</b>
PMTE V.7. “I do not know what to do to turn students on to mathematics.”*	3.74	.79	7	26	68

Note: Percentages reported are the percentage related to the given item, subscale, or total.

**Table 4: PTs’ *Efficacy in Mathematics Teaching* [MTOE] Beliefs Significantly Lower Than *What and How Mathematics Should be Taught* [BT] and *Mathematics in the Real World* [BRW] Beliefs throughout Teacher-Preparation Programs [TPP]**

	Survey Iteration/Time	Mean	Paired Sample t-test		
			Mean difference	T	(p)
MTOE Beliefs	Beginning of TPP (A1)	3.30			
	After Methods Course (A2)	3.44			
	End of TPP (A3)	3.58			
Compare to BT Beliefs	Beginning of TPP (A1)	3.48	.18	2.55	<b>.013</b>
	After Methods Course (A2)	3.89	.45	5.43	<b>&lt;0.001</b>
	End of TPP (A3)	3.79	.20	2.19	<b>.035</b>
Compared to BRW Beliefs	Beginning of TPP (A1)	3.81	.51	6.54	<b>&lt;0.001</b>
	After Methods Course (A2)	4.17	.73	8.78	<b>&lt;0.001</b>
	End of TPP (A3)	4.28	.69	7.56	<b>&lt;0.001</b>

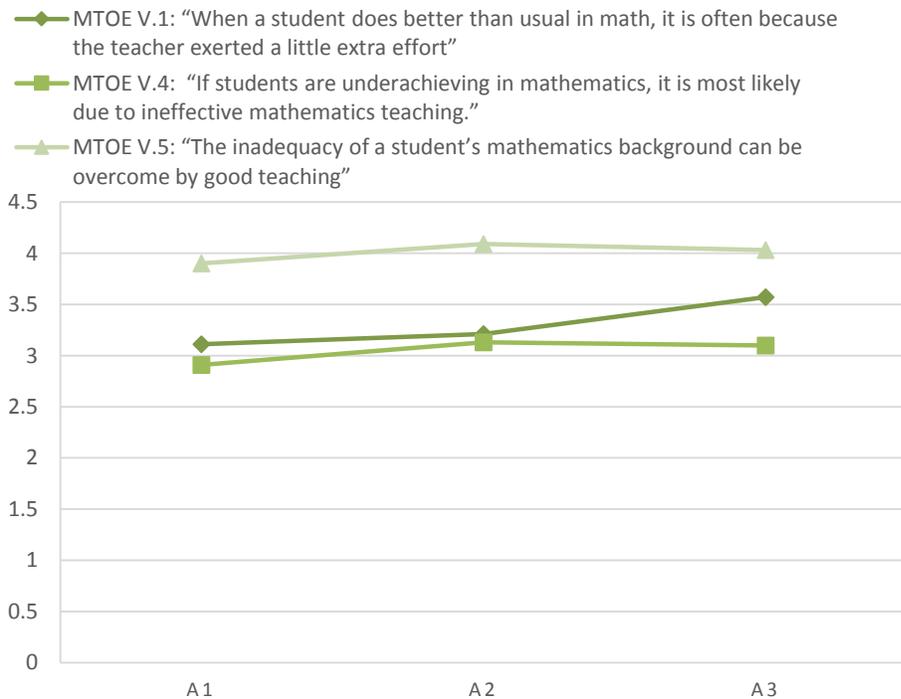
\*Be careful not to make comparisons between survey iterations using this table because varying n-values; rather looking at comparison between types of beliefs at each given time.

**Table 5: Reporting Change in *Efficacy in Mathematics Teaching* [MTOE] Beliefs Using T-Tests (Imputed Values; n=68)**

Likert item	Survey Iteration/Time	Mean	SD/SE	Paired Sample t-test t	(p)
<b>V.1. “When a student does better than usual in mathematics, it is often because the teacher exerted a little extra effort” (MTEBI, #1).</b>	Beginning of TPP (A1)	3.11	.97/.12		
	After Methods Course (A2)	3.21	.94/.11	-.67	.508
	End of TPP (A3)	3.57	.80/.10	-4.04	<0.001
	Change from A1 to A3			-3.34	.001
<b>V.4. “If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching” (#7).</b>	Beginning of TPP (A1)	<b>2.91</b>	.94/.11		
	After Methods Course (A2)	<b>3.13</b>	1.00/.12	-1.47	.147
	End of TPP (A3)	<b>3.10</b>	.83/.10	.25	.804
	Change from A1 to A3			-1.84	.07
<b>V.5. “The inadequacy of a student’s mathematics background can be overcome by good teaching” (#9).</b>	Beginning of TPP (A1)	3.90	.80/.10		
	After Methods Course (A2)	4.09	.80/.10	-1.83	.071
	End of TPP (A3)	4.03	.56/.07	.57	.573
	Change from A1 to A3			-1.437	.155

Note: When the Wilcoxon test was run, same results related to significance found.

**Figure 1: Response Means Per Question Over Time**



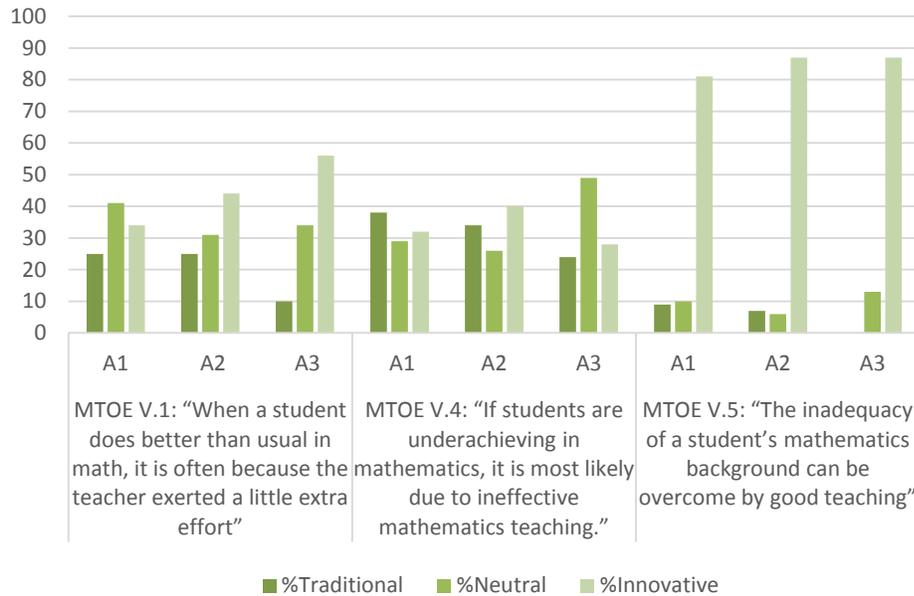
**Table 6: Reporting Change in *Efficacy in Mathematics Teaching* [MTOE] Beliefs Using Frequency Percentages (Imputed Values; n=68)**

Likert item	Survey Iteration/Time	% Leaning Traditional (<2.5)	% Neutral (≥2.5 & <3.5)	% Innovative (≥3.5)
<b>V.1. “When a student does better than usual in mathematics, it is</b>	Beginning of TPP (A1)	25	41	34
	After Methods Course (A2)	25	31	44

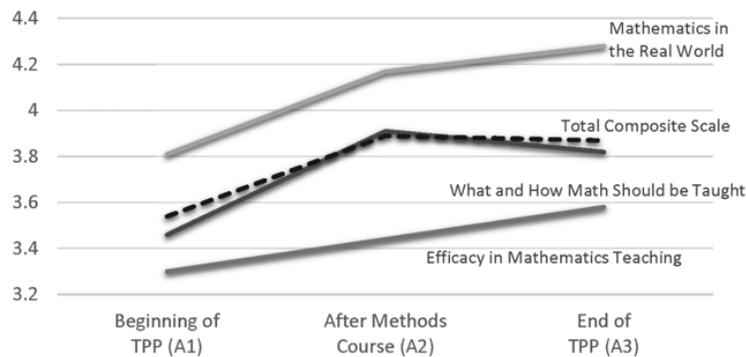
<b>often because the teacher exerted a little extra effort” (MTEBI, #1).</b>	End of TPP (A3)	10	34	56
<b>V.4. “If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching” (#7).</b>	Beginning of TPP (A1)	38	29	<b>32</b>
	After Methods Course (A2)	34	26	<b>40</b>
	End of TPP (A3)	24	49	<b>28</b>
<b>V.5. “The inadequacy of a student’s mathematics background can be overcome by good teaching” (#9).</b>	Beginning of TPP (A1)	9	10	81
	After Methods Course (A2)	7	6	87
	End of TPP (A3)	0	13	87

Note: Imputation of missing data resulted in some non-whole number values.

**Figure 2: Frequency % Per Question Over Time**



**Figure 3: Change in Total and Subscale Beliefs over Time (only using those PTs that responded to all 3 survey iterations; n=36)**



## Set #2: Personal Efficacy in Mathematics Teaching [PMTE]

Though Efficacy in Mathematics Teaching [MTOE] beliefs were the least innovative, PTs' Personal Efficacy in Mathematics Teaching [PMTE] was ranked as the second most innovative subscale (see Table 3). Specifically, by the end of their programs [A3], PMTE responses were significantly more innovative than MTOE responses ( $md=.31$ ,  $t=2.54$ ,  $p=.015$ ). This was not consistent when comparing general normative beliefs about What and How Mathematics Should be Taught [BT] to personal claims about how and what mathematics PTs claimed to teach (ie, My Teaching Practices [P]) at the completion of their programs (Lloyd & Howell, accepted for publication 2018). Personal claims were relatively innovative (see Table 3) and more closely related to their BT counterparts.

The most innovative Personal Efficacy in Mathematics Teaching [PMTE] claim was that PTs could "continuously find better ways to teach mathematics" (V.2, see Table 3). The remaining PMTE item means were greater than 3.70; response percentages leaning traditional did not exceed 9%, and percentages leaning innovative were 63 or greater (see Table 3).

Interestingly, while 84% responded negatively to, "V.6 When a student has difficulty understanding a math concept, I am usually at a loss as to how to help the students understand it better," only 63% agreed to, "V.3 I know the steps to teach concepts effectively." It seems likely that if a PT can help a student "at a loss," then s/he also knows how to teach mathematics effectively. Variation may have to do with confidence in teaching whole-class, initial instruction versus remediating on the individual level. Further research is needed to determine this.

Nonetheless, a majority of PTs reported a strong sense of personal efficacy related to teaching mathematics. This confidence – both in content and pedagogical knowledge – is supported within the open-ended responses as well (see Appendix B). Counting coding revealed that all 39 responders on A3 answered "Yes" to "...hav[ing] a better understanding of the content ... required to teach." When asked if they knew "... teaching strategies to teach ... in depth," 26 out of 34 answered "Yes," and six reported "Yes and no" because "you can always learn more." In particular, a majority commented that they were initially unaware of the complexities of teaching, particularly related to accommodating for all learners, predicting where their students may struggle, accessing innovative resources, and ultimately ensuring "students have a deep understanding." Following their internships, however, they felt much more confident based on their new understanding that innovative instruction is more complex than the traditional teacher-tell method, admitting that they must work "much harder" compared to simply "being able to tell rules." In fact, on A3, when asked how well they could "... explain the concepts as opposed to just the rules," 28 responded either "very well" or "fairly well." While teacher-preparation programs have been criticized for being ineffective (Levine, 2006), review of the Personal Efficacy in Mathematics Teaching [PMTE] findings indicates that they can contribute positively to PTs' personal teaching efficacy.

## Discussion

The data revealed that PTs leaned toward innovative beliefs related to What and How Mathematics Should be Taught [BT] and Mathematics in the Real World [BRW] and claimed to be innovative in their own practices [P and PMTE], but their general beliefs in the efficacy of mathematics teaching (MTOE) were

consistently the lowest. Though Wheatley (2002) contends that lower efficacy beliefs may encourage life-long learning and improvements to teaching (in Siwatu & Chesnut, 2015), an aggregate review of the findings conveys a narrative more similar to his findings related to overconfidence and loci of responsibility.

Many of the open-ended comments, consistent with the literature (Handal, 2003), indicated that being able to enact innovative teaching practices was contextual, based on students, classroom environment, and content. While research repeatedly reports that external conditions constrain teachers (Buehl & Beck, 2015), acting on such beliefs ubiquitously has consequences. PTs may agree with innovative practices in theory (seen by the BT and BRW percentages leaning toward innovation), but may claim: “If it is ‘boring’ content, I can’t make it engaging”; “If not a ‘conducive’ environment, I can’t utilize innovative strategies”; “If my students are too rambunctious or not at grade level, I can’t use these strategies”; “If students learn to hate mathematics through the media, family, and friends, how can I make it exciting,” essentially excusing themselves from committing to such practices based on external constraints.

PTs not only indicated that forces beyond their control were responsible for constraining the use of innovative practices and positive student disposition toward mathematics, they claimed that negative student outcomes were beyond their control. Likert and open-ended responses revealed that many PTs attributed gains in student learning to quality instruction but did not attribute poor learning outcomes to ineffective teaching. In most instances, student failure was attributed to students. Deficit perspectives, whether the genetic-deficit perspective attributing failure to a child’s biological make-up or the cultural-deficit perspective attributing failure to a child’s family and cultural upbringing (Jacob & Jordan, 1993), incorrectly suggest that failure is pre-determined, implying that there is nothing for the teacher to do to prevent it. The consequence of such beliefs leads to instruction that is neither excellent nor equitable, despite practice claims [P] suggesting otherwise<sup>6</sup>.

In summary, with close inspection, (1) mid to high Personal Efficacy in Mathematics Teaching [PMTE] claims, (2) low MTOEV.4 (“If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching.”), and (3) high MTOEV.5 (“The inadequacy of a student’s mathematics background can be overcome by good teaching”) means and frequencies imply the following narrative: “Teachers can overcome students’ mathematical inadequacies. I am one of those teachers: I am doing things the right way; I know how to help my students. And so, when my students do not perform well, it is not my fault.” The narrative is enhanced by open-ended responses, indicting students and other influences beyond the PTs’ control for student failure. Given this narrative, why would there be an expectation that these PTs would sustain innovative, equitable practices, necessary to adequately prepare students for STEM professions?

## Conclusion

External influences can constrain effective, innovative, equitable teaching and, subsequently, positive learning outcomes for all students. As teaching shortages increase, external constraints along with

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<sup>6</sup> What one claims to do does not always align with what they actually do (Gibbard, 1996; Buehl & Beck, 2015). Therefore, further research is needed to examine if PTs observably implement the innovative instructional strategies for which they claim [P].

increased responsibilities are likely to increase, making the profession more challenging for those who remain. It is paramount that these constraints do not serve as licenses to abandon the implementation of innovative teaching. Teacher educators and K-12 educators have parts to play.

Teacher educators must assist PTs in acknowledging that external factors contribute to teaching, so much so that sometimes ideal teaching is inhibited. Acknowledgement is important so that PTs understand that they may not be able to fully enact all ideal practices immediately within constraining contexts and can work toward increased implementation as they grow their voices within school communities. Such acknowledgment may curb the burnout felt by high-quality teachers who leave the profession after a few years (Lloyd, 2012). So that acknowledgment does not result in the abandonment of innovative practices, teacher educators need to teach PTs to assess all students for strengths as well as weaknesses, so not to see students through deficit lenses and to allow strengths to play a role in improving weaknesses.

Similarly, K-12 educators must acknowledge that ideal teaching may be constrained but not entirely abandon innovative pedagogical practices. They must model pedagogical practices that identify students' individual strengths and weaknesses toward targeting learning for all individuals to grow, instead of toward the success of some and predetermined failure of others based on perceived deficits out of educators' control.

K-12 educators should model how to use their voices and teacher educators must help empower PTs to find their voices so all educators, novice and veteran, can comfortably and confidently advocate for what they need to enact innovative practices. Finally, and admittedly lofty, teacher educators need to help mitigate the forces that constrain innovative instruction by encouraging constituents and advocates of public school systems to listen to what teachers need and support them in meeting such needs, acknowledging the challenges of teaching, respecting their professional autonomy and trusting their professional knowledge.

## References

- Avalanche Consulting*. (May, 2016). *Talent Demand Study (sponsored by the Charleston Metro Chamber and Charleston Regional Development Alliance)*. Retrieved from <http://www.charlestonchamber.net/talentedemandstudyupdate/>
- Bandura, A. (1997). *Self-efficacy: The exercise of control*. New York, NY: W.H. Freeman and Company.
- Bikkar, R. S., Beamer, J. E., & Lundberg, I. (1993). Role of mathematics self efficacy in the structural model of mathematics achievement. *Journal of Educational Psychology, 85*(1), 41-48.
- Buehl, M. M., & Beck, J. S. (2015). The relationships between teachers' beliefs and teachers' practices. In H. Fives & M.G. Gill (Eds.), *International handbook of research on teachers' beliefs* (48-65). New York, NY: Routledge.
- Cross Francis, D., Rapacki, L., & Eker, A. (2015). The individual, the context, and practice: A review of the research on teachers' beliefs related to mathematics. In H. Fives & M.G. Gill (Eds.), *International handbook of research on teachers' beliefs* (48-65). New York, NY: Routledge.

- Gibbard, A. (1996). Review essays: thought, norms, and discursive practice: commentary on Robert Brandom, making it explicit. *Philosophy and phenomenological research*, 56 (3), 699–717.
- Green, J., Willis, K., Hughes, E., Small, R., Welch, N., Gibbs, L., & Daly, J. (2007). Generating best evidence from qualitative research: the role of data analysis. *Australian and New Zealand Journal of Public Health*, 31(6), 545-550.
- Handal, B. (2003). Teachers' mathematical beliefs: A review. *The Mathematics Educator*, 13(2).
- Jacob, E., & Jordan, C. (1993). *Minority Education: Anthropological Perspectives. Social and Policy Issues in Education*. Norwood, NJ: Ablex Publishing Company.
- Levine, A. (2006). *Educating school teachers*. Washington DC: The Education Schools Project.
- Lloyd, M. E. (2012). Leaving the Profession: The Context behind One Quality Teacher's Professional Burnout. *Teacher Education Quarterly*, 39(4), 139-162.
- Lloyd, M. E. R., & Howell, M. *Positioning Pre-service Teacher Beliefs along the Traditional-Reform Continuum: An Examination of Normative Beliefs and Discursive Claims*. Manuscript accepted for publication in *The Mathematics Enthusiast* vol. 15, no.3 [July 2018].
- Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: A sourcebook of new methods* (2<sup>nd</sup> ed.). Thousand Oaks, CA: Sage.
- National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Pan, D. (2017, March 23). Report: Lowcountry students are unprepared for region's tech and manufacturing jobs. *The Post and Courier*. Retrieved from [http://www.postandcourier.com/news/report-lowcountry-students-are-unprepared-for-region-s-tech-and/article\\_8cab52ae-0fdd-11e7-aa92-f349634b74f3.html](http://www.postandcourier.com/news/report-lowcountry-students-are-unprepared-for-region-s-tech-and/article_8cab52ae-0fdd-11e7-aa92-f349634b74f3.html)
- Peugh, J. L., & Enders, C. K. (2004). Missing data in educational research: A review of reporting practices and suggestions for improvement. *Review of Educational Research*, 74(4), 525-556.
- Pigott, T. D. (2001). A review of methods for missing data. *Educational Research and Evaluation*, 7(4), 353-383.
- Protheroe, N. (2008). Teacher efficacy: What is it and does it matter? *Principal*, 87, 42-45.
- Rethlefsen, A. L., & Park, H. (2011). A Mixed-Method Study: Assessing the BAR Model's Impact on Preservice Teachers' Efficacy Beliefs. *School Science and Mathematics*, 111(3), 102-117.
- Richardson, G.M., & Liang, L.L. (2008). The use of inquiry in the development of preservice teacher efficacy in mathematics and science. *Journal of Elementary Science Education*, 20(1), 1-16.

Siwatu, K. O., & Chesnut, S. R. (2015). The career development of preservice and inservice teachers: Why teachers' self-efficacy beliefs matter. In H. Fives & M.G. Gill (Eds.), *International handbook of research on teachers' beliefs* (48-65). New York, NY: Routledge.

Will, M. (2016). Analysis projects widening gaps in teacher supply. *Education Week*, 36(5), 1-11.

### Appendix A: Survey

#### I. Background Information

Name: \_\_\_\_\_ Date: \_\_\_\_\_  
 Race/Ethnicity/Gender: \_\_\_\_\_  
 Certification area(s) desired: \_\_\_\_\_  
 Content area(s) for which you feel most comfortable teaching: \_\_\_\_\_  
 Content area(s) for which you feel least comfortable teaching: \_\_\_\_\_  
 Content area(s) for which you feel most comfortable learning: \_\_\_\_\_  
 Content area(s) for which you feel least comfortable learning: \_\_\_\_\_  
 Grade(s) taught during clinical internship (along with any other experiences teaching mathematics – give grade and math subject area if you taught a specific math content area such as Pre-Algebra, Algebra I, Geometry): \_\_\_\_\_

#### II. Practices [P]

Based on your clinical-internship experience and other field experiences, please indicate the degree to which you agree or disagree with each statement by circling the appropriate letters.

	<b>SD</b> Strongly Disagree	<b>D</b> Disagree	<b>N</b> Uncertain or Neutral	<b>A</b> Agree	<b>SA</b> Strongly Agree			
1. I like assigning problems that can be solved in multiple ways (McDougall, 2004, #1).				SD	D	N	A	SA
2. Often I have students complete relevant problems of interest (#2).				SD	D	N	A	SA
3. I provide time and encourage students to share their differing strategies for completing the same problems (#3).				SD	D	N	A	SA
4. Usually, it is not very productive when my students work together (#6).				SD	D	N	A	SA
5. "Every student should feel that mathematics is something he or she can do" (#7).				SD	D	N	A	SA
6. I encourage students to use multiple representations or alternative resources (i.e., manipulatives, technology, etc.) to communicate their mathematical ideas to me and their peers (#10).				SD	D	N	A	SA
7. On graded tasks, I put more emphasis on correct answers than on the process to get to an answer (#11).				SD	D	N	A	SA
8. On non-graded tasks, I put more emphasis on correct answers than on process (#11).				SD	D	N	A	SA
9. Instead of answering students' math questions, I ask them additional questions to help them reason through their initial question (#14).				SD	D	N	A	SA
10. I do not like to assign open-ended tasks because I am concerned that I will not cover the material for a unit in the designated time.				SD	D	N	A	SA
11. I do not like to assign open-ended tasks because I worry that I may not be prepared for unpredictable results (#15).				SD	D	N	A	SA
12. I prefer that my students master basic procedures before tackling complex problems (#16).				SD	D	N	A	SA
13. "I teach students how to communicate their mathematical ideas" (#17).				SD	D	N	A	SA

14. I frequently have to remind my students that a lot of what we learn in mathematics is no much fun, of interest, or relevant to their lives, but that it is important to learn anyway (#20).		SD	D	N	A	SA
15. “When preparing lessons, I generally follow the textbook and/or the proscribed curriculum” (Adamson, Burtch, Cox, Banks, Judson, & Lawswon, n.d., #14) .		SD	D	N	A	SA
16. “When preparing lessons, I generally modify the textbook approach and supplement it with additional problems and/or activities” (#14).		SD	D	N	A	SA
17. “I mainly see my role as a facilitator. I try to provide opportunities and resources for my students to discover or construct concepts for themselves” (#10).		SD	D	N	A	SA
18. “I mainly see my role as a transmitter of knowledge. I try to assist students in arriving at a point of independence and mastery from which they can proceed on their own” (#10).		SD	D	N	A	SA

Please feel free to make additional comments about your practices in the space provided:

### III. Beliefs about How Mathematics Should Be Taught (BT)

1. “Mathematics is computation” (Beswick, Watson, & Brown, 2006, Table 4, #1).		SD	D	N	A	SA
2. Mathematics teachers should be fascinated with how students think and intrigued by their alternative strategies (#3).		SD	D	N	A	SA
3. It is an efficient way to facilitate student mathematical learning by telling students answers (#4).		SD	D	N	A	SA
4. Having students experience slight frustration and tension when solving a problem can be beneficial – even necessary – for learning to occur (#5).		SD	D	N	A	SA
5. The best method for teaching mathematical concepts is an expository style (i.e., demonstrating, explaining, describing, providing examples) (#6).		SD	D	N	A	SA
6. Mathematical concepts need to be presented in the correct sequence (#7).		SD	D	N	A	SA
7. “Ignoring the mathematical ideas that students generate themselves can seriously limit their learning” (#8).		SD	D	N	A	SA
8. Justification of mathematical ideas and statements is an important part of mathematics (#9).		SD	D	N	A	SA
9. To be an effective teacher of mathematics, one must enjoy learning and doing mathematics (#10).		SD	D	N	A	SA
10. An attitude of inquiry should be developed through the teaching of mathematics (#12).		SD	D	N	A	SA
11. Grade-nine mathematics is best taught to groups which are heterogeneous based on ability (#13).		SD	D	N	A	SA
12. “The most important part of instruction is the content of the curriculum” (Adamson, Burtch, Cox, Banks, Judson, & Lawswon, n.d., #11).		SD	D	N	A	SA
13. “The most important part of instruction is that it encourages sense-making or thinking. Content is secondary” (#11).		SD	D	N	A	SA
14. Students must have opportunities to work together to get an in-depth understanding of the content (#13).		SD	D	N	A	SA
15. Working together is problematic because a teacher cannot assess what each individual understands and oft one student does a majority of the work (#13).		SD	D	N	A	SA

Please feel free to make additional comments about your beliefs about how mathematics should be taught in the space provided:

#### IV. Beliefs about the Nature of Mathematics in the Real World (BRW)

1. To be an intelligent consumer, one must be numerate (Beswick, Watson, & Brown, 2006, Table 3, #1).		SD	D	N	A	SA
2. Understanding mathematics is increasingly important in totals society (#4).		SD	D	N	A	SA
3. To function in today's society, being numerate (having "quantitative literacy") is equally as necessary as being literate (#5).		SD	D	N	A	SA
4. Mathematics is necessary to understand media claims (#7).		SD	D	N	A	SA
5. "Mathematics is not always communicated well in the media" (#9).		SD	D	N	A	SA
6. Often people use mathematics in their daily decisions (#10).		SD	D	N	A	SA

Please feel free to make additional comments about your beliefs about mathematics in the real world in the space provided:

#### V. Efficacy – Based on your clinical internship and other teaching experiences (MTOE #1,4,5; PMTE: #2,3,6,7)

1. "When a student does better than usual in mathematics, it is often because the teacher exerted a little extra effort" (Enochs, Smith, & Huinker, 2000, MTEBI, #1).		SD	D	N	A	SA
2. "I am continuously finding better ways to teach mathematics" (#2).		SD	D	N	A	SA
3. "I know the steps to teach mathematics concepts effectively" (#5).		SD	D	N	A	SA
4. "If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching" (#7).		SD	D	N	A	SA
5. "The inadequacy of a student's mathematics background can be overcome by good teaching" (#9).		SD	D	N	A	SA
6. "When a student has difficulty understanding a mathematics concept, I am usually at a loss as to how to help the students understand it better" (#19).		SD	D	N	A	SA
7. "I do not know what to do to turn students on to mathematics" (#21).		SD	D	N	A	SA

Please feel free to make additional comments about your effectiveness as a mathematics teacher in the space provided:

#### VI. Open-ended Questions

Please answer the following questions.

1. Do you feel as though you have a deep understanding of the content you are required to teach?
2. What about your content do you need to learn more about in order to help your students achieve a deep understanding?
3. Do you feel as though what and/or how you are required to teach (including the scope and sequence) can result in a depth of knowledge for your students in this content area? Can you provide any examples?
4. Do you feel as though you know the teaching strategies to teach your students in depth in this content area? Provide examples.
5. If yes to the previous question, do you feel as though you are able to utilize these strategies in order to help your students gain depth in this content area?
6. "How well do you think you can explain the concepts [in this content area] as opposed to just the rules or procedures?" (Stowalter, 2005, #2)
7. Mathematics curricula and teaching in this country is said to be "a mile wide and an inch deep." Do you agree with this? Why or why not?
8. Describe a typical day of teaching in your classroom.
9. Is this how you want to be teaching? Do you feel effective in your teaching? Why or why not?
10. If you answered yes to the previous question, what supports are in place for you to teach how you want to teach and achieve this effectiveness?
11. If you answered no, what constraints can you identify to why you can't teach how you want to be teaching?
12. "To what extent do you feel responsible for your students' learning?" (Showalter, 2005, #1a)
13. "Do you think students are excited about mathematics?" (Stowalter, 2005, #6a)
14. "Why do you think students should take mathematics?" (Stowalter, 2005, #7a)

## Appendix B: Open-ended Response Frequencies

<b>Question: “Do you feel as though you have a better understanding of the content you will be required to teach related to mathematics?” [Efficacy related to content knowledge.]</b>			
Categorized Responses	A1 (n=80)	A2 (n=63)	A3 (n=39)
“No, not yet”	13	11*	
“Yes” with qualifiers (“can learn more”)	67	52	39**
Notes: * Three said “No” on A1. Two originally said, “yes, but it depends” and six originally said “yes” they were comfortable before taking coursework. In fact, one of the “no” responders on A2 said, “Yes, math is my strongest area” on A1. By A3, she stated, “Yes,” suggesting tenuous confidence. ** One respondent said that she had a deep understanding of the content, but “I did not before clinical internship” (did not attribute her deep understanding to coursework). Interestingly, on A2 after her coursework, this PT wrote, “Yes, I have a deep understanding but can always learn more.” She seems to have forgotten the impact her coursework had on her content knowledge by the end (a year later).			
<b>Question: “Do you feel as though you know the teaching strategies to teach your students in depth in this content area?” [Efficacy related to pedagogical knowledge.]</b>			
Categorized Responses	A1 (n=76)	A2 (n=61)	A3 (n=34)
“No.”	64	4 (all responded no earlier)	2 (answered yes in the former iterations, but once faced with reality of teaching, said no)***
“Yes and no.” (“Could always learn more”)		13	6
“Yes.”	12 (learned from former teachers)	43****	26
Notes:*** “No, ... for example, teaching regrouping in second grade...we are not allowed to teach the algorithm so it was new for me to only use models.” Interestingly, in both ME courses, a considerable amount of time was devoted to teaching operations without relying on the standard algorithm. **** Comfortable w/manipulatives, discussions, interdisciplinary lessons, inquiry, student-invented strategies, student reasoning, assessments, collaboration, differentiated instruction, problem-based instruction, and productive struggle			
<b>Question: “How well do you think you can explain the concepts as opposed to just the rules or procedures?” (Stowalter, 2005, #2).</b>			
Categorized Responses	A1 (n=73)	A2 (n=61)	A3 (n=32)
Not very well	39	10***** (five from A1; four of these originally said “fairly well”)	4 (all originally said well or fairly well on A1 – reduction of efficacy) *****
Fairly well	21	31*****	8 (four decreased in efficacy; only one improved from “not very well”)
Very well: “as is” or “I review the content and do a good job”	13	19	20
Unsure		1	
Notes: ***** Five from A1; four of these originally said “fairly well” – realizing that after taking the ME courses, there is more to effective teaching than traditional teach-tell instruction (so a good thing that they “regressed” – as the literature points out not all lowered confidence is bad) ***** One PT specifically answered on A1, “very well,” on A2, “I still need to work on some conceptual understanding but I’ve got a semi-firm grasp,” and on A3, “Poorly.” ***** Attributing to ME courses, saying, “better than a year ago”			
<b>Question: “Do you think students are excited about mathematics?” (Stowalter, 2005, #6a).</b>			
Categorized Responses	A1 (n=76)	A2 (n=59)	A3 (n=34)
Yes	3	5	9
Depends on the teacher	11	18	15

Depends on the student	19	15	4
Students start formal education excited	10	2	3
Very few	12	0	2
No	15	17	1

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## Inspection Worthy Mistakes: Which? And Why?

Carefully select and leverage student errors for whole-class discussions to benefit the learning of all.

Angela T. Barlow, Lucy A. Watson, Amdeberhan A. Tessema, Alyson E. Lischka, and Jeremy F. Strayer

Jana recently attended a ten-day professional development workshop during which she learned about the importance of viewing students' mistakes as learning opportunities (Boaler 2016). Throughout the workshop, Jana and her colleagues celebrated their mathematical mistakes and not only corrected the mistakes but also learned from them. In this setting, she was introduced to the motto, "Mistakes are expected, inspected, and respected" (Seeley 2016, p. 26), which she planned to use during the upcoming school year.

When the school year started, Jana introduced her students to the motto and began reinforcing a positive view of mistakes as learning opportunities. In enacting the motto, though, she faced a new dilemma: Which mistakes should the class inspect? Should the class inspect *all* mistakes? Or are some mistakes more "inspection worthy" than others?

Jana's dilemma is not uncommon. As we become aware of the role of mistakes in learning, we desire to feature students' mistakes in class discussions. Often, discussing a student's mistake provides an opportunity to *critique the reasoning of others*, which is part of the Common Core Standards for Mathematical Practice (SMP 3, CCSS 2010, p. 6–7), particularly when mistakes are not limited to computational errors. Further, inspecting mistakes opens a space for enhancing students' conceptual understandings (Boaler 2015; Borasi 1996).

As Jana previously asked, though, which mistakes are most appropriate for class inspection? With this question in mind, the purpose of this article is to support the reader in selecting mistakes that can be leveraged to benefit the learning of all students. Specifically, we focus on *which* and *why*: *which* mistakes to inspect and *why* these mistakes are inspection worthy. In the next section, we introduce types of mistakes along with ideas to consider when deciding whether a mistake is worthy of class inspection. Then we apply these ideas to a scenario taken from Jana's classroom.

### Which mistakes and why

In considering *which* mistakes to inspect and *why*, we looked at student work from different lessons across different grade levels, focusing on the mistakes that were made and whether they were featured in whole-class discussions. From this process, we identified three types of errors along with ideas to consider when deciding whether inspecting an error will benefit all learners. Before describing the types of mistakes, though, we share two key ideas that arose during this process: The first involved what constitutes a

## Guiding questions to determine which procedural errors are inspection worthy

1. Is the error pervasive or fairly common throughout the class? For example, consider a classroom of third-grade students who are practicing subtraction with three-digit numbers involving regrouping. As the teacher circulates around the room, she notices several students making the procedural error of subtracting the minuend from the subtrahend in the ones place (see **fig. 1a**). Given that several students have made this procedural error, it is worthy of class inspection.
2. Is the error in line with the lesson's goals? Suppose, for example, a fifth-grade class has been studying multiplication with decimals when the teacher notices that a student has incorrectly placed the decimal in the product (see **fig. 1b**) by "lining up the decimals" as if it were an addition problem. Although the error is not necessarily pervasive, correct placement of the decimal in the product represents a fundamental component of the lesson's goals and is, therefore, worthy of inspection.

FIGURE 1

Because several students made the same procedural error, the teacher considered (a) a pervasive procedural error worthy of inspection. When a procedural error represents a fundamental component of the lesson goal, as in (b), we can consider it inspection worthy.

(a)

$$\begin{array}{r} 384 \\ - 276 \\ \hline 112 \end{array}$$

(b)

$$\begin{array}{r} 2.4 \\ \times .7 \\ \hline 16.8 \end{array}$$

which procedural errors are inspection worthy, we offer two guiding questions along with examples (see the **sidebar** above).

### Why?

Inspecting procedural errors that are pervasive and/or aligned with lesson goals offers the class the opportunity to not only identify and correct the error but also justify the reasoning behind correct procedures. By making connections between procedures and their underlying mathematical reasons, students have a chance to “focus attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas” (Stein et al. 2000, p. 16). In doing so, those students who made the error receive support while all students’ understanding is enriched through the sharing of mathematical justifications.

### Inappropriate solution processes

Our second type of mistake involves the solution processes for word problems. Often in these instances, an inspection of the computations alone may not reveal a mistake. That is, the computations may be correct. However, considering the computations in relation to the problem context reveals the error, in that the processes represented by the computations are not appropriate for the problem and represent

mistake. From our viewpoint, a mistake is not limited to a computational error. Rather, mistakes include mathematical thinking, answers, and strategies that are either incorrect or unjustifiable. The second idea involved the mathematical goals, which serve as a lens through which to view all errors. Specifically, throughout our discussion, whether or not we explicitly state so, the mathematical goals of the lesson and/or learning trajectory should be in the foreground of selecting mistakes for inspection. With these ideas in mind, in the following sections, we describe *which* mistakes and *why*.

### Procedural errors

Procedural errors include mistakes in algorithms or other routine procedures. Sometimes procedural errors can be insignificant. For example, a student may write, “ $3 \times 4 = 11$ .” Although correcting his or her mistake is important for the student in this scenario, discussing what seems to be a trivial mistake is unlikely to enhance the mathematical development of all learners. Other procedural errors, though, can potentially enhance all students’ mathematical development and are, therefore, worthy of class inspection. To aid in identifying

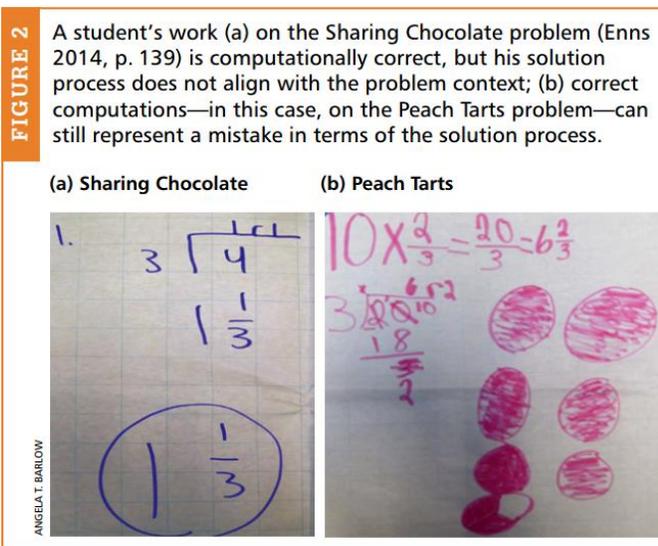
faulty reasoning or a misunderstanding regarding the problem context. Here, we present two examples to illustrate this type of mistake.

### The Sharing Chocolate problem

The Sharing Chocolate problem (Enns 2014, p. 139) reads, in part, as follows:

Two groups of friends are sharing chocolate bars. Each group wants to share the chocolate bars fairly so that every person gets the same amount and no chocolate remains. In the first group of friends, four students receive three chocolate bars. How much chocolate did each person get in the first group?

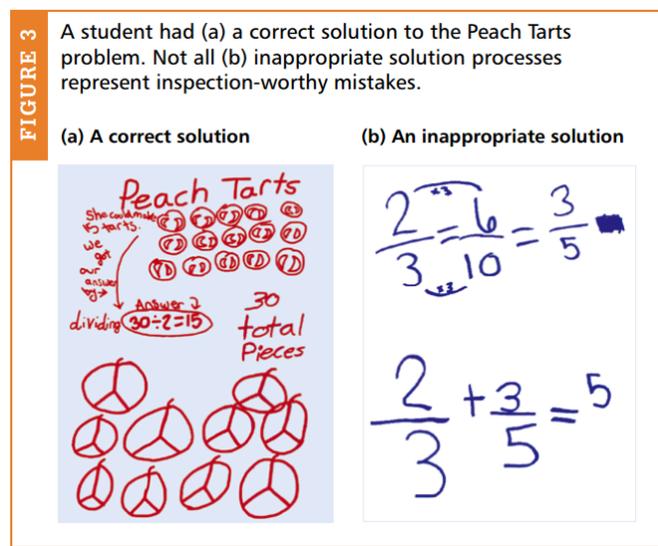
Without consideration of the problem, the student's work (see **fig. 2a**) is computationally correct. The mistake is recognized, though, when one considers the problem. That is, the student's mistake is with his solution process, which does not align with the problem context.



### The Peach Tarts problem

Consider the work in our second example (see **fig. 2b**). The student has correctly multiplied ten by two-thirds and represented her answer with a model. Now contemplate this work in light of the problem the student was solving:

Ms. Stangle wants to make peach tarts for her friends. She needs two-thirds of a peach for each tart, and she has 10 peaches. What is the greatest number of tarts that she can make with 10 peaches? (Chapin, O'Connor, and Anderson 2003, p. 31)



Is  $10 \times \frac{2}{3}$  the appropriate process to use when solving this problem? Actually, the Peach Tarts problem is a division problem: The goal is to determine how many two-thirds are in ten. Therefore, the correct solution process for this problem involves either computing  $10 \div \frac{2}{3}$  or developing a representation (e.g., a picture or concrete manipulatives) that embodies ten divided by two-thirds (see **fig. 3a**). As a result, although the work in **figure 2b** is computationally correct, it does not align with the problem and is, therefore, a mistake representing an inappropriate solution process.

Not all inappropriate solution processes represent inspection-worthy mistakes, though. Consider the work in **figure 3b**, where the student has performed a variety of computations in an attempt to “do something” with the numbers. In talking with the student privately, the teacher found that the reasoning behind the computations was unrelated to the problem context. As a result, inspection of this mistake would likely focus on trying to understand why the student performed the various computations and the errors in them rather than the mathematics represented within the problem context. Therefore, this discussion would not deepen the class’ understanding of the problem context and is not inspection worthy.

### Why?

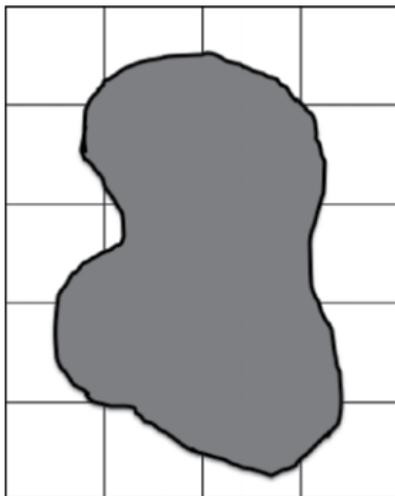
In **figures 2a** and **b**, the mistakes represent opportunities to engage students in reasoning about key mathematical ideas represented within problem contexts. In the Peach Tarts problem, all students would likely benefit from discussing the problem aspects that indicate that it is, in fact, a division problem rather than a multiplication problem. Similarly, a discussion of the mistake in **figure 2a** would provide all students with a meaningful opportunity to assess the reasonableness of this answer. In general, “understanding occurs as a by-product of solving problems and reflecting on the thinking that went into those problem solutions” (Lambdin 2003, p. 11). Reflecting on the mistakes contained within these problem solutions serves to enhance students’ understandings (Boaler 2015). As a result, errors that represent inappropriate solution processes are inspection worthy.

**FIGURE 4**

Some problems typically reveal a view or opinion that students have based on their previous misunderstandings or wrong thinking.

#### The Spilled-Juice Problem

On his homework, Jesse needed to find the perimeter and area of a rectangle. Unfortunately, he spilled his juice on the rectangle (see image below). Help Jesse find the area and perimeter of the rectangle.



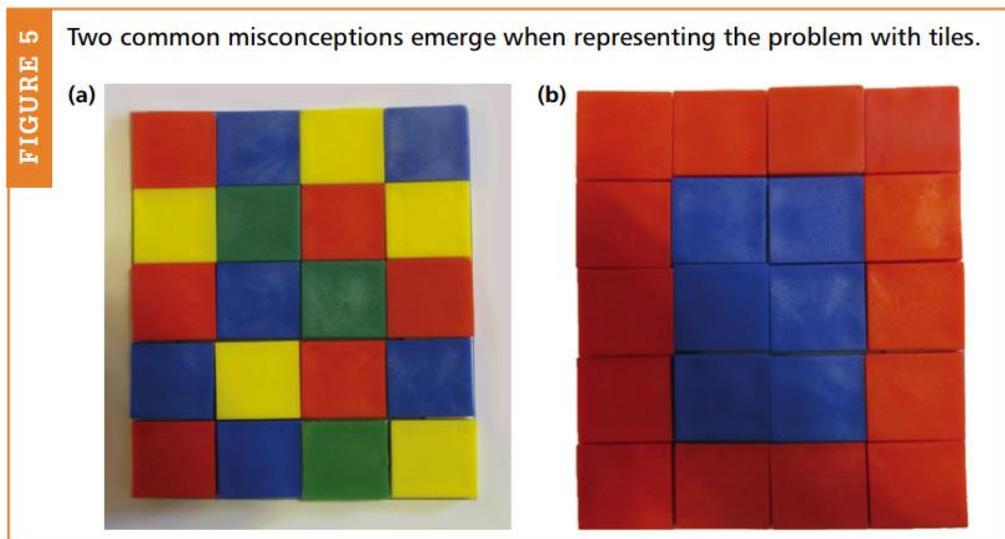
ANGELA T. BARLOW

### Misconceptions

Mistakes that reveal students’ mathematical misconceptions are also worthy of class inspection. We define a misconception as a view or opinion that students mistakenly hold that is based on their previous misunderstandings or wrong thinking. For instance, consider the Spilled Juice problem (see **fig. 4**). Third-grade students normally cover the rectangle with square tiles, (see **fig. 5a**) and count the tiles to find the perimeter and area. To aid in this discussion of mistakes, we provide **figure 5b** as well, which shows a color-coded arrangement of the tiles. Once covered, two common mistakes representing mathematical misconceptions arise as students count the tiles:

1. To find perimeter, students often mistakenly count the “border tiles” (in red in **fig. 5b**). Whether students report the perimeter to be fourteen (a literal count of the border tiles) or eighteen (double-counting the corners), counting the squares represents a misconception regarding perimeter. That is, the students do not recognize perimeter as a measurement of length that should be found by counting the units of length (i.e., the sides of the squares) that make up the perimeter of the rectangle.

- In determining area, students will state that the area is represented by the “inside squares” and then limit the area to the enclosed blue squares (see **fig. 5b**). This misconception regarding what constitutes the area of a figure is likely related to viewing the border tiles as representing the perimeter.



### Why?

The two featured mistakes represent fundamental misconceptions regarding the lesson’s mathematical goals (i.e., perimeter and area) and are, therefore, worthy of class inspection. By discussing these mistakes, students have the opportunity to grapple with the underlying concepts; in this case, what are perimeter and area, and how are they measured?

Explicit confrontation of preconceptions or misconceptions creates cognitive dissonance in which students begin to question and rethink their preconceptions, and further instruction and reflection can now help students understand the new concept. (Tobey and Fagan 2013, p. 181)

Inspecting mistakes that represent misconceptions can support all students in either *correcting* or *refining* their understandings of the concepts. Therefore, mistakes that involve fundamental misconceptions related to the lesson’s mathematical goals are worthy of inspection.

### Revisiting Jana’s dilemma

With a focus on *which* and *why*, we now consider Jana’s dilemma from the opening scenario. In an introductory lesson on area, Jana posed the Quilt task (see **fig. 6**) to her students. Her goal was to support students in seeing area as the amount of space covered by a figure. Jana gave students copies of the task and a set of pattern

**FIGURE 6** In posing the Quilt task, Jana’s goal was to support students in seeing area as the amount of space covered by a figure.

**The Quilt Task**

Adrianna is trying to decide which quilt she should buy for her bed. Each quilt was created using pattern blocks. Adrianna wants to choose a quilt that covers the greatest area. Should she choose Quilt A or Quilt B?

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blocks. As students worked, Jana circulated around the classroom, inquiring about students' strategies. Jana's dilemma of which mistakes were inspection worthy arose as she noted three mistakes (see **table 1**).

**TABLE 1** Walking around the classroom and asking students about their strategies brought Jana's dilemma to the forefront.

**Summary of mistakes for the Quilt task**

Student	Description of strategy	Additional information
<b>Emily</b>	Covered the quilts with triangles and then miscounted; concluded that Quilt B with 24 triangles was larger than Quilt A with 22 triangles.	Emily's mistake of miscounting the blocks was unique to Emily.
<b>Caroline</b>	Covered the quilts with an assortment of blocks and then counted; concluded that Quilt B with 12 blocks was larger than Quilt A with 7 blocks.	Caroline's mistake of counting blocks without consideration for their different sizes was a common mistake throughout the class.
<b>Ryan</b>	Covered the quilts with an assortment of blocks and then counted; concluded the quilts covered the same area because both had 15 blocks.	Ryan's mistake was basically the same as Caroline's, with the exception that his led to the correct answer (i.e., the quilts covered the same area).

In considering these errors, Jana recognized Emily's mistake as a procedural error because she had miscounted. Because this error was not pervasive, Jana chose not to have the class inspect it. In thinking about Caroline's and Ryan's mistakes, Jana noted that the two errors represented the same inappropriate solution process, that is, counting the blocks as if each block covered the same amount of area. She chose to have the class inspect Caroline's mistake (see **fig. 7**) and used the Imagine the Alternative strategy (Rathouz 2011). The following discussion ensued.

**Teacher:** So, without thinking about whether the answer is right, let's look at this covering of the quilts. Think for a moment, how might a student use this arrangement of the blocks to argue that quilt B is larger than quilt A? Trey?

**Trey:** Maybe it's 'cause quilt B has more blocks.

**Teacher:** Would someone like to talk some about this idea? Lizzy?

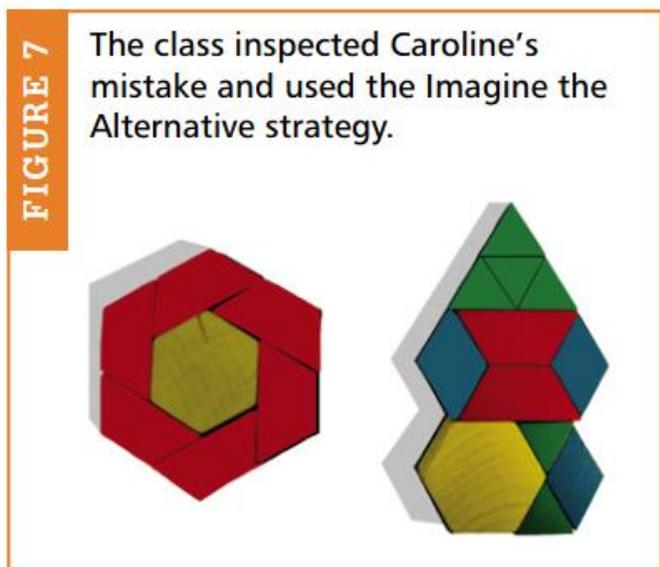
**Lizzy:** B has twelve blocks, but A has only seven blocks.

**Manuel:** I don't think you can just count the blocks.

**Teacher:** Why do you say that, Manuel?

**Manuel:** 'Cause, like, two green blocks is not bigger than one yellow block.

**Teacher:** Hmmmm, that's an interesting point. Let's all think about that for a moment.



As students discussed this mistake, fundamental concepts related to the lesson's focus of area began to arise, providing the cognitive dissonance needed to disrupt students' current thinking so that the focus might shift toward more productive ways of comparing area.

## Conclusion

The inspection of mistakes can play a powerful role in an individual's learning process (Boaler 2015). In our own lessons, we have noted that students who might be hesitant to share their own ideas are often willing to inspect other's mistakes. Through discussions of inspection worthy mistakes, errors can be leveraged to benefit the learning of *each and every* student. By focusing on *which* and *why*, our students benefit from expecting, inspecting, and respecting mistakes.

## References

- Barlow, Angela T., Natasha E. Gerstenschlager, and Shannon E. Harmon. 2016. "Learning from the Unknown Student." *Teaching Children Mathematics* 23, no. 2 (September): 337–44.
- Boaler, Jo. 2015. *Mathematical Mindsets: Unleashing Students' Potential through Creative Math, Inspiring Messages, and Innovative Teaching*. San Francisco, CA: Jossey-Bass.
- Borasi, Raffaella. 1996. *Reconceiving Mathematics Instruction: A Focus on Errors*. Issues in Curriculum Theory, Policy, and Research Series. Norwood, NJ: Ablex.
- Bray, Wendy S. 2013. "How to Leverage the Potential of Mathematical Errors." *Teaching Children Mathematics* 19, no. 7 (March): 424–31.
- Chapin, Suzanne H., Catherine O'Connor, and Nancy Canavan Anderson. 2003. *Classroom Discussions: Using Math Talk to Help Students Learn (Grades 1–6)*. Sausalito, CA: Math Solutions Publications.
- Common Core State Standards Initiative (CCSSI). 2010. *Common Core State Standards for Mathematics (CCSSM)*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.  
[http://www.corestandards.org/wpcontent/uploads/Math\\_Standards.pdf](http://www.corestandards.org/wpcontent/uploads/Math_Standards.pdf)
- Enns, Ed, ed. 2014. "Problem Solvers: Problem: Sharing Chocolate." *Teaching Children Mathematics* 21, no. 3 (October): 136–39.
- Lambdin, Diana V. 2003. "Benefits of Teaching through Problem Solving." In *Teaching Mathematics through Problem Solving: Prekindergarten–Grade 6*, edited by Frank K. Lester Jr. and Randall I. Charles, pp. 3–14. Reston, VA: National Council of Teachers of Mathematics.
- Pace, Michelle H., and Enrique Ortiz. 2016. "Get the Goof." *Teaching Children Mathematics* 23, no. (October): 138–44.

Rathouz, Margaret. 2011. "3 Ways That Promote Student Reasoning." *Teaching Children Mathematics* 18, no. 3 (October): 182–89.

Seeley, Cathy L. 2016. *Making Sense of Math: How to Help Every Student Become a Mathematical Thinker and Problem Solver*. Alexandria, VA: ASCD, copublished with the National Council of Supervisors of Mathematics and the National Council of Teachers of Mathematics.

Stein, Mary Kay, Margaret Schwan Smith, Marjorie A. Henningsen, and Edward A. Silver. 2000. *Implementing Standards Based Mathematics Instruction: A Casebook for Professional Development*. New York: Teachers College Press.

Tobey, Cheryl R., and Emily R. Fagan. 2013. *Uncovering Student Thinking about Mathematics in the Common Core, Grades K–2: 25 Formative Assessment Probes*. 1st ed. Thousand Oaks, CA: Corwin.

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